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Anomalous exponent in the solution of a nonlinear diffusion equation

Tao Tu[1], **G Cheng**[1] **and Hua Sheng**[2]

[1] Laboratory of Quantum Communication and Quantum Computation, University of Science and Technology of China, Hefei, Anhui, 230026, People's Republic of China
[2] Department of Physics, University of Maryland, College Park, MD 20742, USA

E-mail: tutao@mail.ustc.edu.cn

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Abstract
We show with a broad class of nonlinear diffusion equations that renormalization group (RG) theory can be used to understand the conventional asymptotic analysis. The long time behaviour of this system and the anomalous exponent can be derived with the RG equation.

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1. Introduction

Renormalization group (RG), and in particular its quantum field theory implementation has provided us with essential tools for the description of phase transitions and critical phenomena beyond mean field theory [1]. Some years ago, it was found that there were also important applications in non-equilibrium phenomena and asymptotic analysis [2]. In particular, applications to calculate the anomalous exponents in the asymptotic behaviour of nonlinear partial differential equations were considered by Goldenfeld and his colleagues, in the case of Barenblatt's equation [3], the modified porous medium equation [4] and the turbulent-energy-balance equation [5].

In this short paper we apply the RG method to a specific class of nonlinear diffusion equations such as $\partial_t u(x,t) - \frac{1}{2}\partial_x^2 u(x,t) = f\left(x, u, \partial_x u, \partial_x^2 u\right)$, where the nonlinearity term is expressed as $\varepsilon x^m u^n (\partial_x u)^p \left(\partial_x^2 u\right)^q$ kept in one-space dimension, and m, n, p, q are integers that satisfy $n+p+q = 1$, $p+2q-m = 2$, so that the equation is dimensionally correct without introduction of a new time or space scale [13]. This type of nonlinearity term describes a variety of physical phenomena [13]: from basic thermodynamics a number of key nonlinearities of the above form can be derived because of the limitations of the linear theory of diffusion [6, pp 55–68]. It also arises from inhomogeneities in the diffusion coefficient in the flux or the variable-dependent potentials in Fick's law [7, chap. 11]. Particular examples involve $u^{-1}u_x^2$ which models inverse temperature-dependent heat diffusion and phase transition [8]

$xu^{-2}(\partial_x u)^3$, $x^2u^{-2}(\partial_x u)\left(\partial_x^2 u\right)^2$, which arise in phase transitions involving alloys [9]. Other applications are the filtration of a compressible fluid through an elastic porous medium, which is irreversibly deformable [10], and magnetic fields with permeability depending on the field strength [11, chap. 6], etc.

The purpose of this paper is to add another example to substantiate the point of view that the fundamental hypothesis of intermediate asymptomatic behaviour can be derived with the RG process. An outline of this paper is as follows: in section 2 we show the discussion of this system in the standard asymptotic analysis, in section 3 we use the RG approach to obtain the anomalous asymptotic behaviour and there the anomalous exponent appears naturally, and in section 4 a brief summary and outlook are given.

2. Self-similarity of the second kind: asymptotics of this system

We consider the equation in the case when the initial condition is given by

$$u(x,0) = \frac{m_0}{\sqrt{2\pi l^2}} \exp\left(-\frac{x^2}{2l^2}\right) \tag{1}$$

and we seek solutions which vanish at infinity.

To illustrate how the anomalous exponent appears in this system naturally, let us begin with a very simple and familiar example: the one-dimensional diffusion equation $\partial_t u(x,t) - \frac{1}{2}\partial_x^2 u(x,t) = 0$ with the initial condition (1). The long-time behaviour of the problem is $u \sim \frac{m_0}{\sqrt{t}} \exp\left(-\frac{x^2}{2t}\right)$, i.e., this system has the conventional asymptotics of the first kind [12].

For convenience, we select the following special but nontrivial example of the original equation

$$\partial_t u(x,t) - \tfrac{1}{2}\partial_x^2 u(x,t) = \varepsilon u^{-1}(\partial_x u)^2 . \tag{2}$$

At first glance the desired asymptotic solution of equation (2) must be expressed in the form

$$u = \frac{m_0}{\sqrt{t}}\phi(\xi,\varepsilon) \qquad \xi = \frac{x}{\sqrt{t}}. \tag{3}$$

However, this leads to difficulty. In order to see this, we integrate equation (2) with respect to x from $-\infty$ to $+\infty$ to obtain the following relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} u(x,t)\,\mathrm{d}x = \varepsilon \int_{-\infty}^{+\infty} \frac{[u_x(x,t)]^2}{u}\,\mathrm{d}x. \tag{4}$$

Substituting equation (3) into equation (4), we have

$$0 = \varepsilon\frac{m_0}{t}\,[1 + O(\varepsilon)] \tag{5}$$

and for $\varepsilon \neq 0$ this system evidently has no nontrivial solution for an expression such as equation (3). Thus, the assumption of self-similarity of the first kind as (3) turns out to be incorrect.

Now we make a more complicated assumption for the asymptotic behaviour as suggested by Barenblatt [12], i.e., self-similarity of the second kind:

$$u = \frac{m_0}{t^{\frac{1}{2}+\alpha}}\phi(\xi,\varepsilon) \qquad \xi = \frac{x}{\sqrt{t}}. \tag{6}$$

To show that the anomalous exponent α does exist in a nontrivial result, we perform an ε-expansion and restrict ourselves to the first term of this expansion:

$$\alpha = \beta\varepsilon + O(\varepsilon^2)$$
$$\phi(\xi) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\xi^2}{2}\right) + O(\varepsilon). \tag{7}$$

Substituting equation (7) into equation (4), we have

$$\alpha = -\varepsilon + O(\varepsilon^2). \tag{8}$$

3. Anomalous exponent calculation by the RG method

In a previous paper Caginalp has discussed the long-time behaviour of the original equation by defining an appropriate similarity-group transformation and seeking fixed points [13]. In this paper we will use the RG method in Gell-Mann and Low type to determine the anomalous self-similar exponent.

To investigate the actual form of the solution of the nonlinear diffusion equation, we construct a perturbation in ε. The formal solution to the equation with the initial condition (1) is

$$u(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}y\, G(x-y,t)u(y,0) + \varepsilon \int_0^t \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}y\, G(x-y,t-s) y^m u^n (\partial_y u)^p \left(\partial_y^2 u\right)^q \tag{9}$$

where G is the Green's function of the operator $\partial_t u - \frac{1}{2}\partial_x^2 u$:

$$G(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right). \tag{10}$$

Introducing the expansion

$$u(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \cdots \tag{11}$$

we find that the zeroth-order term is simply

$$u_0 = \frac{m_0}{\sqrt{2\pi(t+l^2)}} \exp\left(-\frac{x^2}{2(t+l^2)}\right). \tag{12}$$

The first-order term is calculated in a straightforward manner using the zeroth-order solution. We find that

$$u_1 = \int_0^t \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}y \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(x-y)^2}{2(t-s)}\right) y^m \frac{m_0}{\sqrt{2\pi(s+l^2)}} \times \exp\left(-\frac{y^2}{2(s+l^2)}\right) \left(-\frac{y}{s+l^2}\right)^p \left[\left(\frac{y}{s+l^2}\right)^2 - \frac{1}{s+l^2}\right]^q. \tag{13}$$

As anticipated, u_1 diverges as $t \to \infty$ or, equivalently, as $l \to 0$ [2]. We are interested in the behaviour at small l. Using $\omega = \frac{y}{\sqrt{s+l^2}}$, we obtain

$$u_1 = \frac{m_0}{2\pi\sqrt{t}} \exp\left(-\frac{x^2}{2t}\right) \int_0^t \mathrm{d}s \frac{1}{s+l^2} \times \int_{-\infty}^{+\infty} \mathrm{d}\omega\, (-1)^p \omega^{p+m} (\omega^2 - 1)^q \exp\left(-\frac{\omega^2}{2}\right) + O(l^2) \tag{14}$$

so that the singular part of u_1 is given by

$$u_{1s} = \frac{m_0}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)\left[-\beta \ln\left(\frac{t}{l^2}\right)\right] \tag{15}$$

where

$$\begin{aligned} -\beta &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\omega (-1)^p \omega^{-2n} (\omega^2 - 1)^q \exp\left(-\frac{\omega^2}{2}\right) \\ &= \sum_{j=0}^{q} (-1)^{j+p} \binom{q}{j} 1 \times 3 \times \cdots \times |2p + 4q - 2j - 3| \end{aligned} \tag{16}$$

where $\binom{q}{j}$ is a binomial term. Thus, the bare perturbation theory result is

$$u(x,t) = \frac{m_0}{(2\pi t)^{1/2}} \exp\left(-\frac{x^2}{2t}\right)\left[1 - \varepsilon\beta \ln\left(\frac{t}{l^2}\right) + O(\varepsilon^2)\right] + \text{regular terms.} \tag{17}$$

The quantity m_0, the 'initial mass', cannot be obtained from knowledge of $u(x, t)$ at large times; therefore m_0 is considered to be 'unobservable' at large times in the same way that the 'bare' electric charge is unobservable at long distances according to quantum electrodynamics [14]. We cure the logarithmic divergence of the bare perturbation series by introducing the renormalized mass

$$m_0 = Z^{-1}\left(\frac{l}{\mu}\right) m_R \tag{18}$$

where μ is an arbitrary length to let Z be dimensionless. The renormalization constant Z is introduced to absorb the divergences as $l \to 0$, and so depends upon l and ϵ. We proceed by assuming the expansion

$$Z\left(\frac{l}{\mu}\right) = 1 + a_1\left(\frac{l}{\mu}\right)\varepsilon + \cdots. \tag{19}$$

The coefficients a_n should be chosen to cancel the divergence in $u(x, t)$ as $l \to 0$, order by order in ε. To $O(\varepsilon)$, we obtain

$$a_1\left(\frac{l}{\mu}\right) = \beta \ln\left(\frac{C_1\mu^2}{l^2}\right) \tag{20}$$

where C_1 is an arbitrary dimensionless number. Hence we have

$$u_R(x,t) = \frac{m_R}{(2\pi t)^{1/2}} \exp\left(-\frac{x^2}{2t}\right)\left[1 - \varepsilon\beta \ln\left(\frac{t}{C_1\mu^2}\right) + O(\varepsilon^2)\right]. \tag{21}$$

This expression shows that u_R remains finite as $l \to 0$, because l does not enter it at all. In fact, relation (21) describes a family of solutions. We choose a particular solution by requiring that at some given time t^*, $u_R(0, t^*)$ has the value U:

$$u_R(0, t^*) = U. \tag{22}$$

Then the corresponding solution to order ε is

$$u_R = U\left(\frac{t^*}{t}\right)^{1/2} \exp\left(-\frac{x^2}{2t}\right)\left[1 - \varepsilon\beta \ln\left(\frac{t}{t^*}\right) + O(\varepsilon^2)\right]. \tag{23}$$

This expression will be referred to as the renormalization expansion.

The actual solution does not depend upon the choice of the time t^*. Using the renormalization group argument of Gell-Mann and Low [14], we get

$$\frac{\mathrm{d}u_R}{\mathrm{d}t^*} = \frac{\partial u_R}{\partial t^*} + \frac{\partial u_R}{\partial U}\frac{\partial U}{\partial t^*} = 0. \tag{24}$$

This equation is analogous to the renormalization group equation in field theory. We can evaluate $t^*\frac{\partial U}{\partial t^*}$ perturbatively from the expression u_R. We obtain

$$t^*\frac{\partial U}{\partial t^*} = -U\left(\tfrac{1}{2} + \varepsilon\beta + O(\varepsilon^2)\right). \tag{25}$$

Solving this differential equation for U, we have

$$U = A(t^*)^{-\left(\frac{1}{2}+\varepsilon\beta+O(\varepsilon^2)\right)} \tag{26}$$

where A is a constant of integration determined by the initial conditions. We insert this value into equation (23) and set $t^* = t$ because t^* can be selected in an arbitrary way. Hence we obtain the final representation:

$$u_R = \frac{A}{t^{\frac{1}{2}+\alpha}} \exp\left(-\frac{x^2}{2t}\right) \tag{27}$$

where $\alpha = \varepsilon\beta$ is the anomalous exponent in the solution of the nonlinear diffusion equation in the intermediate asymptotic region by RG approach to the first order of ε. Choosing $m = 0, n = -1, p = 2$ and $q = 0$ corresponding to the nonlinearity term $u^{-1}(\partial_x u)^2$, the expression (16) gives the anomalous exponent $\alpha = -\varepsilon$, which is the result given by equation (8) in section 2.

4. Conclusion

To summarize, we calculate the anomalous exponent by the RG method in the case of a specific class of nonlinear diffusion equations. This exponent is similar to the anomalous dimension in field theory and may be calculated by using the RG method instead of the asymptotic analysis technique. It is, however, clear from this paper that this does not restrict the general nature of the method for other evolution equations. The RG method developed by Goldenfeld and his colleagues has been successfully applied by many authors to quite a wide class of problems (for a recent brief summary, see, e.g., [15]), and we favour attempts in this direction.

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